PHYSICAL REVIEW VOLUME 132, NUMBER 3 1 NOVEMBER 1963

Self-Consistent Deviations from Unitary Symmetry*

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(Received 24 June 1963)

A method of investigating the possible dynamic origin of symmetries among the strong interactions is illustrated by application to a model with vector mesons that are self-consistently bound states of one another. The SU_3 model, with eight vector mesons, is concentrated upon. All possible types of first-order perturbations are treated in the ladder approximation, and some second-order effects are also considered. The results emerging from a qualitative discussion uniquely suggest the possibility (in addition to the degenerate mass solution) of a self-supporting small mass splitting structure of the type leading to the Gell-Mann–Okubo mass formula. Moreover, SU_2 symmetry is necessarily retained, although the differentiation between charge and hypercharge is not possible in a theory which does not include electromagnetism.

I. INTRODUCTION

THERE are many indications that SU_3 symmetry provides a useful way to correlate the properties of the strongly interacting particles.[1–7] Although the departures from perfect symmetry are large, they have themselves a characteristic structure exhibited through the retention of isotopic spin symmetry and through the Gell-Mann–Okubo formula for the mass differences.[8–12] One of the central questions of strong interaction physics is whether these characteristic relations among the masses and coupling constants must be obtained from *ad hoc* postulates, or whether they arise automatically through the interworkings of dynamical effects.[13,14,14a] We consider here a restricted aspect of this question using a simplified treatment. We shall discuss interrelations among the various manifestations of a small dissymmetry, and shall suggest a possible dynamical reason for the particular form which the dissymmetry takes.

Charge and hypercharge conservation, as well as charge conjugation symmetry, are assumed to hold. We do not examine perturbations leading to the possibility that these concepts break down, because they seem to have validity outside the domain of the strong interactions.

In the investigation of this problem, we have been forced, as a result of the limitations in our present understanding of strong interaction dynamics, to confine our attention to a qualitative discussion of simple models having the following general features: (1) We consider only states with two relatively light particles; (2) we use the ladder approximation; (3) we introduce an over-all cutoff Λ, which we adjust to enable the equations for the masses and coupling constants to be self-consistently satisfied, but which we do not attempt to calculate from the particle size implied by our model; (4) we study only those solutions of the nonlinear self-consistency equations in which the dissymmetries are small.

We shall examine in this paper a simple model in which only vector mesons appear. It has been shown, for this model, that if the masses are equal, the mesons must fit into the adjoint representation of some semi-simple Lie group.[9] In our discussion of the possibility of nonequal masses, we consider primarily the case of eight vector mesons, which corresponds to SU_3 being taken as a first approximation.

By considering this simple vector meson model, we will be able to describe the underlying physical concepts as well as the calculational techniques in the simplest situation possible. A remarkable limitation on the possible types of dissymmetry arises from our

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14a *Note added in proof.* After we submitted this paper, we learned of a paper by E. Abers, F. Zachariasen, and G. Zemach [Phys. Rev. **132** (1963), (to be published)] in which the possibility of obtaining interaction symmetries from bootstrap mechanisms was analyzed in detail. The general similarity of outlook in this paper and in ours will be clear to the reader, but we feel that it would be helpful if we take this opportunity to point out two differences of emphasis. First, we have assumed that the homogeneity of the self-consistency equations would preclude the existence of any solution to their approximated versions, unless an adjustable parameter were introduced, and that with a free parameter, there would be a large number of solutions (cf. Refs. 9, 11). Second, we have, in consequence, concentrated upon the delimitation of such solutions, in particular, assuming the existence of a symmetrical solution, we have here examined solutions with broken symmetry, whereas Abers, Zachariasen, and Zemach have focused upon establishment of the plausibility of the existence of symmetrical solutions.

TABLE I. Mass dissymmetry matrices $D(r,T)$ for octuplets.

	Diagonal elements								Off-diag.	Norm.
(Y,T,T_3)	$(1,\frac{1}{2},\frac{1}{2})$	$(1,\frac{1}{2},-\frac{1}{2})$	(0,1,1)	(0,1,0)	(0,1,−1)	(0,0,0)	$(-1,\frac{1}{2},\frac{1}{2})$	$(-1,\frac{1}{2},-\frac{1}{2})$	$(0,T,0)$	
$D(8,0)$	1	1	−2	−2	−2	2	1	1	0	$2\sqrt{5}$
$D(8,1)$	$\sqrt{3}$	$-\sqrt{3}$	0	0	0	0	$-\sqrt{3}$	$\sqrt{3}$	2	$2\sqrt{5}$
$D(27,0)$	3	3	−1	−1	−1	−9	3	3	0	$2\sqrt{30}$
$D(27,1)$	−1	1	0	0	0	0	1	−1	$\sqrt{3}$	$\sqrt{10}$
$D(27,2)$	0	0	1	−2	1	0	0	0	0	$\sqrt{6}$
$D(S)$	1	1	1	−3	1	−3	1	1	0	$2\sqrt{6}$
$D(1,0)$	1	1	1	1	1	1	1	1	0	$2\sqrt{2}$

qualitative discussion of this model, which encourages the belief that the problem of strong interaction symmetry can be solved by considering it as a problem of self-consistency. The discussion of the vector-meson model which is given here can be generalized without difficulty to a more realistic model in which there are boson-baryon couplings, leading to similar results which will be reported in a subsequent paper.

In the generalized vector-meson model (generalized to nonequal masses) the masses and coupling constants are determined by self-consistency equations which could be derived (for example) by the method of Zachariasen and Zemach,[15] or from the Bethe-Salpeter equation,[16] and which are similar to those in Ref. 9. If we eliminate the coupling constants from these equations, the self-consistency equation for the mass matrix $(M^2)_{ab}$ can be written in the form[17]:

$$M^2=k(M^2,\Lambda)\,, \tag{1}$$

which just represents the dependence of the calculated mass on the masses of the particles being bound together and of the exchanged particles.

Following Glashow,[8] we represent M^2 in terms of normalized tensor operators $D(rT)$:

$$M^2=\sum_{rT} a(rT)D(rT)\,, \tag{2}$$

where r denotes a representation of SU_3 and T the total isotopic spin. Since charge Q and hypercharge Y are conserved, the D's correspond to $Q=Y=0$. For the vector mesons, charge-conjugation symmetry and Hermiticity limit r to (2,2) (27-fold), $(1,1)_S$ (8-fold), and (0,0) (singlet). The explicit forms of the $D(rT)$ are given in Table I. The coefficient $a(0,0)$, which is related to the average $\mathfrak{M}^2$, is determined by a choice of scale and will be ignored hereafter. A self-consistent solution which is completely symmetric corresponds to $a(rT)=0$ for $r\neq(0,0)$. We assume that such a solution of (1) exists.

If we restrict our attention to small dissymmetries, we may expand (1) to second order as

$$a(rT)=K(r)a(rT)+\sum_{(r_1T_1,r_2T_2)} L(rT,r_1T_1,r_2T_2)a(r_1T_1)a(r_2T_2)\,. \tag{3}$$

We have denoted by $K(r)$ the eigenvalue of $K_{ab,cd}=\partial k_{ab}/\partial(M^2)_{cd}$ in the representation r. Equation (3), which consists of five simultaneous quadratic equations, has a large number of solutions [not all of which necessarily correspond to solutions of (1)].

We shall classify the solutions of (1) or (3) according to their relation to the root diagram of SU_3 (Fig. 1). In expanding the mass dissymmetry matrix, we have labelled the tensor operators by their isotopic spin. Now, SU_2 is contained as a subgroup in SU_3 in three distinct ways, corresponding to the three sets of reciprocal roots; any of these sets could be interpreted as the isotopic spin operators. If we express the mass matrix corresponding to one of the solutions of (1) in terms of tensor operators labeled according to one of the other two sets of reciprocal roots, we obtain new coefficients $a'(rT)$ which will also correspond to a solution of (1). In other words, the solutions of (1) or (3) form a representation of the symmetry group of the root diagram.

The nonlinear terms of (3) are, in part, determined by the Clebsch-Gordan coefficients of SU_3; in particular, they must be consistent with the subgroups of SU_3. We look especially at dissymmetries which are invariant under a subgroup of SU_3; the direct product of one-dimensional representations of a group is one dimensional, so other types of dissymmetry cannot be mixed in through the nonlinearities. The possibility that all the a's vanish, i.e., that complete symmetry under SU_3 is maintained, is, of course, obtained as a trivial

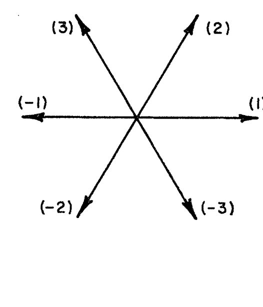

FIG. 1. The root diagram for SU_3.

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[17] We follow the conventional practice in considering the squares of the boson masses to be the variables.

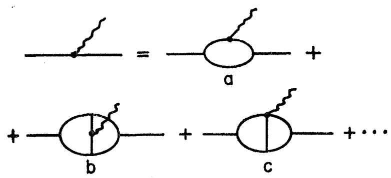

FIG. 2. First-order mass perturbation of vector mesons as an expansion in self-energy graphs. The solid lines represent the vector mesons. The perturbation can act on a propagator or a vertex (see Fig. 3) and it is described by a wriggly line. This suggests its interpretation as an external field with $Q=Y=0$. We wish to emphasize, however, that the "bare coupling constants" of this field need not be considered to be different from zero.

solution of (3). We see that there must also be solutions of (3) [with $a((8),0)$ and $a((27),0)$ being the only nonvanishing coefficients] which maintain isotopic spin symmetry, and further solutions [with $a((27),2)$ also different from zero] which exhibit charge symmetry but not full isotopic spin symmetry. From these solutions, others can be generated by permuting the roots, as described above. The dissymmetry given by

$$D(S)=(\tfrac{1}{3}\sqrt{5})D(27,0)+\tfrac{2}{3}D(27,2) \qquad (4)$$

is invariant under permutation of the roots, and therefore, will necessarily arise as one of the solutions.

II. THE SU_2 MODEL

Before examining the possibility of unsymmetrical solutions in the SU_3 model, we shall illustrate the approach by discussing SU_2, which is much simpler. The only possible dissymmetry in the adjoint representation corresponds to $T=2$: $M_\pm^2=\mathfrak{M}^2+a/\sqrt{6}$, $M_0^2=\mathfrak{M}^2-2a/\sqrt{6}$. Then upon eliminating Λ by requiring that self-consistency be attained with a given fixed value of $\mathfrak{M}^2$, (1) takes the form

$$\begin{aligned} a&=k(a)\\ &=Ka+La^2+\cdots. \end{aligned} \qquad (5)$$

From the truncated expansion one obtains the solutions

$$\begin{aligned} a&=0,\\ a&=(1-K)/L. \end{aligned} \qquad (6)$$

If it should turn out that K were close to unity, the neglect of higher-order terms in (5) would be justified, and we would have found a second self-consistent solution which was slightly unsymmetrical. If, on the other hand, $1-K$ were not small, one would need to look at the exact expression in order to see whether additional solutions existed. We suggest that if the dissymmetry predicted by (6) is large ($a\gg\mathfrak{M}^2$), the existence of an unsymmetrical solution is implausible.

One of the contributions to K is described by graph 2a, which represents the change in the mass of a bound state arising from a change in the mass of one constituent. The charged states are made up of one neutral and one charged meson, and the neutral state of two charged mesons. Therefore, we have

$$\begin{aligned} \Delta_\pm(2a)&=\alpha(\Delta_\pm+\Delta_0)=-\alpha a/\sqrt{6},\\ \Delta_0(2a)&=\alpha(2\Delta_\pm)=2\alpha a/\sqrt{6}, \end{aligned} \qquad (7)$$

where α is a constant, presumably positive, which characterizes the internal structure of the particles. From (7), we obtain $K_{2a}=-\alpha$. We wish to emphasize that the effect represented by Eq. (7) is not to be looked upon as a modification of the ordinary second-order self-energy term, even though we have pictured it by such a graph. In fact, the end vertices in graph (a) in Fig. 2 could be interpreted as standing for ladder graphs with an indefinite number of vertical rungs, in accordance with our view of the vector states as states bound by the ladder-approximation potential.

If the mass of the exchanged quantum is increased by an amount Δ, the potential is presumably decreased in magnitude, leading to a resultant energy increased by $\beta\Delta$. This effect is represented by graph 2(b). Since for SU_2 the exchange quantum has the same charge as the bound state, we obtain $K_{2b}=+\beta$.

We continue to assume, as in Ref. 9, that at every vertex the independent covariants have fixed ratios. Then the question of a change in the ratios of the coupling constants does not arise for SU_2, because there is only one.

The values of α and β cannot be calculated without a detailed dynamical model; in particular, they depend on the way the cutoff is introduced. A simple way is to assume that for each line, the propagator is regularized with auxiliary masses which are taken to be proportional to the physical mass. One then obtains for the ladder approximation, by considering the effect of a change in the unit of mass, the identity

$$2\alpha+\beta=1. \qquad (8)$$

One expects, moreover, that $\alpha\approx\beta$, so that each would be about $\frac{1}{3}$. In the ladder approximation, therefore, one anticipates that $K\approx 0$.

The coefficient L of the second-order term in (5) represents nonlinear effects on the bound-state energy of a change in the masses of the constituents or of the exchanged quantum. The size, and even the sign of these terms appears to be quite model-dependent. We expect, in any event, that $|L|\lesssim\mathfrak{M}^{-2}$. The SU_2 model could not, then, have a solution with a small dissymmetry, and we should like to suggest the possibility that it does not admit any second unsymmetrical solution. A detailed investigation of this question would be of considerable interest.

III. THE SU_3 MODEL: FIRST-ORDER PERTURBATIONS

We now return to the quantity $K(r)$ appearing in (3), and evaluate it within the ladder approximation.

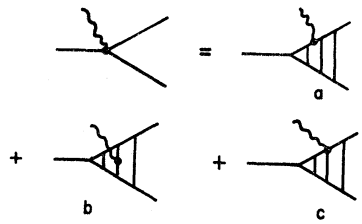

FIG. 3. First-order vertex modifications showing explicitly the use of the ladder approximation. Symmetrization with respect to the three legs is implied.

Graph 2(a) gives the following contribution to $K_{ab,cd}\Delta_{cd}$:

$$\begin{aligned}\Delta_{ab}(2a) &= \alpha F_{cra}F_{sdb}(\delta_{cd}\Delta_{rs}+\delta_{rs}\Delta_{cd}) \\ &= -2\alpha F_{acr}F_{bdr}\Delta_{cd}\,, \end{aligned} \tag{9}$$

where α is the same quantity appearing in (7). The eigenvalues of $F_{acr}F_{bdr}$ are as calculated in Refs. 5, 7, and 10, apart from a different normalization, which we take to be $F_{abr}F_{bas}=\delta_{rs}$ in the present work, as in Ref. 9. One finds that $K_{2a}(8)=\alpha$, $K_{2a}(27)=-\frac{2}{3}\alpha$.

Graph (b) of Fig. 2 leads to:

$$\Delta_{ab}(2b)=\beta F_{rma}(-2F_{rsc}\Delta_{cd}F_{mnd})F_{nsb}\,. \tag{10}$$

The factor (-2) is introduced into (10) so that β will have the same meaning as before. This equation can be written as

$$\Delta_{ab}(2b)=\beta[F_{acr}F_{bdr}+2F_{amr}F_{bnr}F_{mcs}F_{nds}]\Delta_{cd}\,,$$

from which we find $K_{2b}(8)=0$ and $K_{2b}(27)=5\beta/9$. Comparing these results with those obtained in the previous section, we see that the contributions to $K(27)$ are similar to the corresponding ones of the SU_2 model. The additional possibility (8) of the SU_3 model, on the other hand, corresponds to a $K(8)$ which, on the basis of these graphs, is positive. This encourages us to go further with the calculations.

In the SU_3 model there is also a contribution to K from graph 2(c), although the calculation of the vertex modifications described by Fig. 3 requires that, at least implicitly, we go beyond the ladder approximation. It is also, unfortunately, much more difficult. The method by which we shall calculate the change in the coupling constants can be described as follows: The self-energy corrections of Fig. 2 can be thought of as the expectation value of the "mass energy" [Fig. 2(a)] and of the potential energy [Figs. 2(b) and 2(c)] in the unperturbed states. At the same time, we can calculate the admixture of components from the representation (20) $\equiv(3,0)\oplus(0,3)$ into the eigenstates. These admixtures give directly the vertex modifications which are to be used in calculating the perturbed potential. We are only interested in the changes in the ratios of the coupling constants, because their average is determined by the equation for self-consistency of the average of the masses.

We know that the vertex must involve the three lines symmetrically. However, our method of calculation treats them in an unsymmetrical way, because one particle is thought of as a bound state of the other two. This would not matter, if we used the exact potential in our calculation and treated the change in the normalization condition properly, because then the symmetry would arise automatically. In our calculation, we use the ladder approximation and an artificial normalization; to overcome these limitations we must symmetrize explicitly at the end.

We denote by $-V$ the coefficient of the momentum-dependent factors in the one-particle exchange potential. It is assumed that these momentum-dependent factors are approximately the same in all elements of the potential, so that they may be adequately represented by a suitable average. Then we may assume that the mass will be given by an equation having the form

$$M^2=S-f(V)+\alpha A\,, \tag{11}$$

which is a matrix equation in which M^2-S is an eigenvalue and A represents the change in the masses of the bound particles. Since for exact SU_3 symmetry, V has only one nonvanishing eigenvalue, we may write, to a sufficient approximation, $f(V)=Vf(\lambda)/\lambda$, where λ denotes the larger eigenvalue of V. This may be expanded about the unperturbed potential V_0 as $f(V)\approx t(V_0+\delta V)+t'\delta V_d$, where δV_d is the part of the perturbation which is diagonal in the unperturbed representation. We define t so that V_0 has a unit eigenvalue.

We next separate V into two parts, corresponding to the two graphs (b) and (c) of Fig. 2 [as well as to Figs. 3(b) and 3(c)]. We write

$$V=v+\beta B/(t+t')\,,$$

where B is the change in the mass of the exchanged quantum (the coefficient is chosen so that our previous definition of β is retained) and where v depends only on the coupling constants. We, finally, obtain

$$\begin{aligned} f(V)=tv+[t/(t+t')]\beta B+t'(v_d-V_0) \\ +[t'/(t+t')]\beta B_d\,. \end{aligned} \tag{12}$$

In calculating for the (1,1) type of dissymmetry, we look at $D(8,0)$: $\Delta_\rho=-2a'$, $\Delta_\varphi=+2a'$, and $\Delta_M=+a'$ [with $a(8,0)=(2\sqrt{5})a'$]. The modified coupling constants are:

$$\begin{aligned} g(\rho^3)&=g_0(\rho^3)(1+f_0)\,, \\ g(\bar{M}M\varphi)&=g_0(\bar{M}M\varphi)(1+f_1)\,, \\ g(\bar{M}M\rho)&=g_0(\bar{M}M\rho)(1+f_2)\,. \end{aligned} \tag{13}$$

For a dissymmetry described by $D(8,0)$, it can be shown by general arguments that to first order $f_2=0$ and $f_1=-\frac{2}{3}f_0$. We will demonstrate shortly that these ratios form a self-consistent choice.

For the ρ state, we write $|\rho\rangle=\xi|\rho\rho\rangle+\eta|\bar{M}M\rangle$. As a matrix acting on the column vector (ξ,η), we have

$$A=\begin{pmatrix}-4a' & 0 \\ 0 & 2a'\end{pmatrix}. \tag{14}$$

The matrix v can be calculated from the explicit reduction coefficients for SU_3 (we do not give the details here), and has the form:

$$v=\begin{pmatrix} \frac{2}{3}(1+f_0)^2 & \frac{1}{3}\sqrt{2}(1+f_2)^2 \\ \frac{1}{3}\sqrt{2}(1+f_2)^2 & \frac{1}{2}(1+f_1)^2-\frac{1}{6}(1+f_2)^2 \end{pmatrix}. \quad (15)$$

In the present section, only the terms in v which are linear in the f_i are relevant, but we have given the complete form so it can be referred to later. The matrix B is gotten by multiplying each term in v by the mass change of the exchanged particle which is responsible for that term. The result is easily seen to be:

$$B=a'\begin{pmatrix} -\frac{4}{3}(1+f_0)^2 & \frac{1}{3}\sqrt{2}(1+f_2)^2 \\ \frac{1}{3}\sqrt{2}(1+f_2)^2 & (1+f_1)^2+\frac{1}{3}(1+f_2)^2 \end{pmatrix}. \quad (16)$$

As a check, it can be verified that the expectation values of A and B in the unperturbed state agree with the results obtained above.

In v, we introduce the relations between the f_i mentioned following Eq. (13). Then, to first order in f_0 and a', we find the following admixture into the state of the ρ:

$$\langle(20)|\rho\rangle=\sqrt{2}h, \quad (17)$$

where

$$h=2\alpha a'/t+\beta a'/(t+t')+\tfrac{2}{3}f_0. \quad (18)$$

From (17), one easily calculates:

$$\xi=(\tfrac{2}{3})^{1/2}(1+h), \quad \eta=(\tfrac{1}{3})^{1/2}(1-2h). \quad (19)$$

Next, we consider the state $|M\rangle=\xi|\rho M\rangle+\eta|\varphi M\rangle$, for which the matrices are

$$A=\begin{pmatrix} -a' & 0 \\ 0 & 3a' \end{pmatrix}, \quad (14')$$

$$v=\begin{pmatrix} \frac{2}{3}(1+f_0)(1+f_2)-\frac{1}{6}(1+f_2)^2 & \frac{1}{2}(1+f_1)(1+f_2) \\ \frac{1}{2}(1+f_1)(1+f_2) & \frac{1}{2}(1+f_1)^2 \end{pmatrix}, \quad (15')$$

$$B=a'\begin{pmatrix} -\frac{4}{3}(1+f_0)(1+f_2)-\frac{1}{6}(1+f_2)^2 & \frac{1}{2}(1+f_1)(1+f_2) \\ \frac{1}{2}(1+f_1)(1+f_2) & \frac{1}{2}(1+f_1)^2 \end{pmatrix}. \quad (16')$$

Again calculating to first order, we have

$$\langle(20)|M\rangle=h, \quad (17')$$

and

$$\xi=\tfrac{1}{2}\sqrt{2}(1+h), \quad \eta=\tfrac{1}{2}\sqrt{2}(1-h). \quad (19')$$

There is no admixture into the state $|\varphi\rangle=|\bar{M}M\rangle$. In this state, A, v, and B are just numbers:

$$A=2a', \quad (14'')$$

$$v=\tfrac{1}{2}(1+f_1)^2+\tfrac{1}{2}(1+f_2)^2, \quad (15'')$$

$$B=a'[(1+f_1)^2-(1+f_2)^2]. \quad (16'')$$

We now symmetrize the vertices $\langle\rho|\bar{M}M\rangle$ and $\langle M|\rho M\rangle$, as well as $\langle\varphi|\bar{M}M\rangle$ and $\langle M|\varphi M\rangle$, among the three lines. Then we obtain

$$f_0=h, \quad f_1=-\tfrac{2}{3}h, \quad f_2=0. \quad (20)$$

On comparing (20) with (18), we find

$$h=6\alpha a'/t+3\beta a'/(t+t'). \quad (21)$$

When these values are substituted into (15), (15′), or (15″), we obtain from the expectation values

$$K_{2c}(8)=2\alpha'+\beta, \quad (22)$$

where $\alpha'=\alpha(t+t')/t\approx\alpha$.

The calculation of $K_{2c}(27)$ proceeds in the same way as above. In fact, if we considered a perturbation proportional to $D(27,0)$, we could use the same matrices v and merely make appropriate modifications in A and B. However, for later reference, we shall discuss $D(S)$, which is actually simpler. The two particles ($\mathfrak{i}$) with $Q=Y=0$ have $\Delta_{\mathfrak{i}}=-3a'$, and the remaining six ($\mathfrak{o}$) have $\Delta_{\mathfrak{o}}=a'$ (with $a(S)=2a'\sqrt{6}$). There are only two coupling constants:

$$\begin{aligned} g(\mathfrak{o}^2\mathfrak{i})&=g_0(\mathfrak{o}^2\mathfrak{i})(1+f_1), \\ g(\mathfrak{o}^3)&=g_0(\mathfrak{o}^3)(1+f_2), \end{aligned} \quad (23)$$

which are related by $f_2=-3f_1$.

In the state $|\mathfrak{i}\rangle$ there is no admixing: $|\mathfrak{i}\rangle=|\mathfrak{oo}\rangle$. The values of A, v, and B are

$$\begin{aligned} A&=2a', \\ v&=\tfrac{1}{3}(1+f_2)^2+\tfrac{2}{3}(1+f_1)^2, \\ B&=a'[\tfrac{1}{3}(1+f_2)^2-2(1+f_1)^2]. \end{aligned} \quad (24)$$

For the $\mathfrak{o}$ we write $|\mathfrak{o}\rangle=\xi|\mathfrak{oi}\rangle+\eta|\mathfrak{oo}\rangle$. The matrices are

$$A=\begin{pmatrix} -2a' & 0 \\ 0 & 2a' \end{pmatrix},$$

$$v=\begin{pmatrix} \frac{2}{3}(1+f_1)^2 & \frac{1}{3}\sqrt{2}(1+f_1)(1+f_2) \\ \frac{1}{3}\sqrt{2}(1+f_1)(1+f_2) & \frac{1}{3}(1+f_1)^2 \end{pmatrix}, \quad (25)$$

$$B=a'\begin{pmatrix} \frac{2}{3}(1+f_1)^2 & \frac{1}{3}\sqrt{2}(1+f_1)(1+f_2) \\ \frac{1}{3}\sqrt{2}(1+f_1)(1+f_2) & -(1+f_1)^2 \end{pmatrix}.$$

From these we calculate the following admixture:

$$\langle(20)|\mathfrak{o}\rangle=\sqrt{2}h', \quad (26)$$

where

$$h'=4\alpha a'/3t-4\beta a'/9(t+t')+4f_1/9. \quad (27)$$

This leads to

$$\xi=(\tfrac{2}{3})^{1/2}(1+h'), \quad \eta=(\tfrac{1}{3})^{1/2}(1-2h'). \quad (28)$$

Symmetrizing as before, we obtain $f_1=\frac{2}{3}h'$, $f_2=-2h'$, which gives

$$h'=\frac{12\alpha'}{19}\left(\frac{3\alpha}{t}-\frac{\beta}{t+t'}\right). \quad (29)$$

The expectation values of v then lead to

$$K_{2c}(27)=-16\alpha'/3\cdot19+16\beta/9\cdot19.$$

Adding together our estimates of the various contributions to K, we have

$$\begin{aligned} K(8)&=\alpha+2\alpha'+\beta\,,\\ K(27)&=-\tfrac{2}{3}\alpha-(16/57)\alpha'+(111/171)\beta\,. \end{aligned} \quad (30)$$

These formulas should be considered in the light of our estimate that α, α', and β should be near to $\frac{1}{3}$. It is certainly consistent with (30) to have a second type of solution of the self-consistency equation (1) in which the predominate dissymmetry is of the (1,1) type; on the other hand, a (2,2) type of dissymmetry is not favored. We suggest, as the origin of the Gell-Mann–Okubo rule, that

$$|1-K(8)|\ll|1-K(27)|\,. \quad (31)$$

While our calculations indicate the plausibility of obtaining (31) from a more complete theory, we cannot claim to have established it.

IV. SECOND-ORDER TERMS

Before we examine the relative sizes of the different second order terms in (3), we shall point out an important result which follows directly from the assumption that it is the nonlinear terms which determine the magnitudes of small dissymmetries. This result also depends on the dominance of the (1,1) type of dissymmetry. If the suggestion (31) is correct, we are justified in neglecting the terms in (3) referring to the (2,2) dissymmetry when we calculate $a(8,0)$ and $a(8,1)$. Then (3) is reduced to

$$\begin{aligned} (1-K)a(0)&=L[a(0)^2-a(1)^2]\,,\\ (1-K)a(1)&=-2La(0)a(1)\,, \end{aligned} \quad (32)$$

where we have simplified the notation by omitting reference to the representation, and where the ratios of the terms on the right are obtained from Clebsch-Gordan coefficients. The solutions to (32) are

$$a(0)=a(1)=0\,, \quad (33a)$$

$$a(1)=0\,,\quad a(0)=a\equiv(1-K)/L\,, \quad (33b)$$

$$a(0)=-\tfrac{1}{2}a\,,\quad a(1)=\pm\tfrac{1}{2}\sqrt{3}a\,. \quad (33c)$$

Solution (33b) corresponds to retention of isotopic spin symmetry. The two solutions (33c) also correspond to SU_2 symmetry, but with different pairs of the roots shown in Fig. 1 being interpreted as the isotopic-spin displacement operators. This feature of the solutions to (32) is actually a direct consequence of the general discussion given in the introduction, and is accordingly more general than Eq. (32); we only need to assume that the nonlinear terms involving the $a(8,T)$ are more important than those involving the $a(27,T)$.

We now ask, what *a priori* criteria might be used to distinguish among the solutions (33)? We may note that the value of the cutoff parameter Λ which is fixed by the self-consistency requirement will be different for solution (33a) and for the other three; this may ultimately lead to a way to discriminate against (33a). However, the three solutions (33b,c) have completely identical properties as far as the strong interactions are concerned. In other words, there is no way, as long as only the strong interactions are considered, to decide which of the conserved quantities should be called the "charge" and which the "hypercharge." It is the electromagnetic interactions which distinguish between these quantities. The ambiguity is, therefore, a necessary feature of any attempt to derive isotopic spin within the strong interactions; the explanation of the relation between isotopic spin and electromagnetic interactions must lie in the nature of electromagnetism.

In looking more closely at the second-order terms, we are chiefly interested in the following points: (1) We want to know something about the magnitude of the coefficient L in (32); in particular, whether there might be reason to suspect it of being anomalously small. (2) A small (1,1) dissymmetry will induce, in second order, a still smaller (2,2) dissymmetry; choosing $T=0$, we write

$$[1-K(27)]a(27,0)=L'a(8,0)^2\,, \quad (34)$$

under the assumption that second-order terms involving $a(27,0)$ can be neglected. It has been argued that $|1-K(8)|\ll|1-K(27)|$ in order to justify this assumption as well as our more general conclusions. It is clear that the ratio L'/L also has considerable significance. (3) Hitherto, our remarks have been directed towards solutions of Eq. (3) in which the (1,1) dissymmetry predominates. We must also look at other solutions. In fact, Eq. (3) has so many solutions that a complete analysis would require extensive numerical computation. Since we are able to derive, by our present means, only qualitative information about the coefficients, such an investigation would be premature. Nevertheless, we should like to examine at least one example, and the discussion in the Introduction suggests a suitably simple one—that afforded by the dissymmetry $D(S)$; accordingly, we look at the coefficient L'' in the equation

$$[1-K(27)]a(S)=L''a(S)^2\,. \quad (35)$$

We commented in Sec. II that the second-order terms in the SU_2 model could not be easily estimated. There are similar effects here, which we must similarly ignore, but there are, in addition, some relatively simple second order terms, arising from the admixing of the (20) configuration, which can be obtained from Eq. (11). While we would not, perhaps, trust the magnitudes of

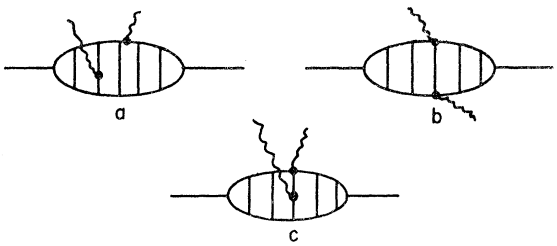

FIG. 4. Some graphs describing second-order perturbations.

the effects so derived, we might have more confidence in their ratios. For a picture of these effects, we refer to Fig. 4.

The graph 4(a), in which there are two perturbations that act in different sections of the ladder, describes an effect we estimate by standard second-order perturbation theory. The change in M^2 is proportional to the squares of the admixture coefficients; for the (1,1), $T=0$ perturbation

$$\Delta_\rho = -2th^2, \quad \Delta_M = -th^2, \quad \Delta_\varphi = 0. \tag{36}$$

There is a change in the average value of M^2 within the multiplet, which, however, is eliminated by a readjustment of the scale. If we express the deviations (36) in terms of the normalized operators $D(r,0)$, we find for the coefficients

$$a_{4a}(8,0) = (4/\sqrt{5})th^2, \quad a_{4a}(27,0) = -(3/\sqrt{30})th^2. \tag{37}$$

For the (S) perturbation, we have

$$\Delta_{i} = 0, \quad \Delta_{j} = -2th'^2, \quad a_{4a}(S) = -th'^2\sqrt{6}. \tag{38}$$

We now substitute into (37) and (38) the values of h and h' previously derived [Eqs. (21) and (29)], and obtain the results

$$\begin{aligned} L(4a) &= (9/5\sqrt{5})x(2\alpha'+\beta)^2, \\ L'(4a) &= -(27/20\sqrt{30})x(2\alpha'+\beta)^2, \\ L''(4a) &= -(36/19^2\sqrt{6})x(3\alpha'-\beta)^2, \end{aligned} \tag{39}$$

where $x = t/(t+t')^2$.

The graph 4(b) is calculated as the expectation values in the unperturbed states of the terms in v which are of second order in the f_i. Similarly, graph 4(c) represents the contribution to the expectation values of B which is of first order in the f_i. The results are given in Table II, along with (39). The parameter y represents $(t+t')^{-1}$. Both x and y are expected to be near to $\mathfrak{M}^{-2}$, or, likely, somewhat smaller.

TABLE II. Second-order perturbation coefficients associated with Fig. 4.

Graph	L	L'	L''
4(a)	$+0.805(2\alpha'+\beta)^2x$	$-0.246(2\alpha'+\beta)^2x$	$-0.041(3\alpha'-\beta)^2x$
4(b)	$+0.224(2\alpha'+\beta)^2y$	$+0.091(2\alpha'+\beta)^2y$	$+0.040(3\alpha'-\beta)^2y$
4(c)	$+0.179\beta(2\alpha'+\beta)y$	$+0.037\beta(2\alpha'+\beta)y$	$+0.115\beta(3\alpha'-\beta)y$

From Table II we see that graph 4(a) seems to be the most important. Note that the individual contributions to L are larger than those to L' or L''. Moreover, there seems to be some tendency toward cancellation among the contributions to L' and L'' which is not evident in L. The nature of these second order results, therefore, reinforces our contention that self-consistent deviations from SU_3 symmetry would necessarily be of the (1,1), or 8-fold, type.

Finally, we wish to remark on the influence of a true external perturbation, such as provided by electromagnetic interactions, on the self-consistent solutions. Let us denote by $\eta(0)$ and $\eta(1)$ the extra self-energy terms which are added to the right hand sides of Eq. (32). Solution (33a) is then perturbed to

$$a(0) = \eta(0)/(1-K), \quad a(1) = \eta(1)/(1-K), \tag{33a'}$$

to first order in the η's, while for (33b) we obtain

$$\begin{aligned} a(0) &= (1-K)/L - \eta(0)/(1-K), \\ a(1) &= \eta(1)/3(1-K). \end{aligned} \tag{33b'}$$

The extra contribution to the large dissymmetry $a(0)$ would, of course, be difficult to verify empirically. The feature of (33b′) which we wish to point out is the greater stability of the unsymmetrical solution against an additional $T=1$ perturbation.

V. SUMMARY

The usual way to discuss the approximate symmetries of strong interactions is in terms of a zero-order symmetrical Hamiltonian and a perturbing addition having a specified structure. In this approach, one has the technical advantage that there is a well-known systematic procedure for deriving the consequences of the initial assumptions. We are now engaged in the construction of a new theory of the symmetries, in which it is assumed that these features do not reflect directly features of the Hamiltonian, but arise as special characteristic simplicities of the lowest lying states. Our method of investigation is to show first that there exists a self-consistent set of particles (in a certain approximation) which exhibits a full symmetry, and then, using this solution as the starting point, to examine the possibility of self-consistent sets of the same particles in which the mass ratios differ from unity. In studying the self-consistency, we trace the influence of a given dissymmetry among the interacting particles upon the coupling constants and masses calculated for the bound states. The technique of calculation in our self-consistency approach is exactly the same as in the standard one—all the familiar machinery of perturbation theory is evoked. Once it is realized that nothing useful of the conventional theory has been lost, there need be no difficulty about accepting the changed starting point.

In ascribing a dynamical origin to the symmetry, however, a great deal is gained; the possible dissymmetries become, so to speak, "quantized" by the self-consistency requirement, which determines not only the qualitative features of the allowable deviations from symmetry, but also their numerical magnitudes. In the model we have studied, in which eight vector mesons interact among themselves, self-consistency has led to a number of interesting results concerning the departure from SU_3 symmetry.

We found, first, that the model is very stable against a perturbation from symmetry which has the transformation properties of a 27-fold tensor, and much less stable against a perturbation of the 8-fold type.[18] This has the consequence that the model can be expected to have additional self-consistent solutions which have a small dissymmetry which is predominantly characterized by an 8-fold tensor, but does not have solutions with a small 27-fold dissymmetry. Since we consider a rather simplified model, and treat it only qualitatively, we do not attempt to calculate the numerical value of the dissymmetry. However, the fact that the magnitudes are determined by self-consistency leads at once, as we have shown, to retention of SU_2 symmetry. In other words, our model leads, in a naturalistic way, both to the Gell-Mann–Okubo mass formula and to the isotopic spin concept.

Finally, it should be pointed out that our present work is limited in three respects. First, we do not have a useful criterion for choosing between the completely symmetrical solution and the solution with perturbed symmetry; in fact, we have not even given an *a priori* reason for preferring SU_3 to any other group. Second, we have relied on qualitative arguments in estimating the parameters which describe the internal dynamical structure of the bound states. We should like to suggest, as a particularly useful program of numerical computation, the precise evaluation of bound state energies for a variety of input masses. This would determine these parameters more exactly, and also allow the exploration of the possibility of very unsymmetrical solutions to Eq. (1). Third, it is clear that the interrelations among the dissymmetries of different kinds of particles will be of particular interest. This last question we intend to discuss further in another paper.

[18] It would be quite wrong to speak of the symmetrical solution as being *unstable* against an 8-fold perturbation, since the magnitude of the deviation is, in fact, prescribed.

PHYSICAL REVIEW VOLUME 132, NUMBER 3 1 NOVEMBER 1963

Singularities of a Relativistic Scattering Amplitude in the Complex Angular Momentum Plane*

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(Received 24 May 1963)

The analytic properties of the partial-wave amplitude in the complex angular momentum plane are investigated in a relativistic scalar-meson theory. Using a N/D decomposition, the numerator and denominator are calculated by perturbative expansions to fourth order in the coupling constant. Higher order poles at negative integer values of l are found in both the numerator and denominator, leading to a breakdown in their perturbation expansions near these singularities. The same breakdown occurs for all but the leading Regge trajectory. It is further found that, to fourth order, due to inelastic processes, the denominator has branch cuts for values of l near negative integers.

I. INTRODUCTION

We present here a relativistic model for scalar-meson–scalar-meson scattering and discuss the analytic properties of the resulting partial-wave scattering amplitude in the complex angular momentum plane. The method consists of decomposing the partial-wave amplitude, $a(s,l)$,

$$a(s,l)=N(s,l)[1+D(s,l)]^{-1}, \qquad (1.1)$$

and calculating both N and D by perturbation expansions using an interaction of the form $(g/3!):\phi^3:$. The details of the method will be given in the next section.

Although the procedure is explicitly carried out to fourth order, we believe that some of the results established hold for all orders in g. One result is that the Regge trajectories

$$l=\alpha_n(s), \quad n=1, 2, \cdots, \qquad (1.2)$$

with the possible exception of $n=1$, cannot be expanded in a power series in g. As we shall show later, this is intimately connected with the failure of the perturbation expansion near negative integer values of l.

Another result obtained is that the contribution of inelastic processes to the scattering amplitude leads, in a simple way, to the existence of branch cuts in the complex l plane for the denominator function to fourth

* Work supported in part by the U. S. Atomic Energy Commission.